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American Options under Stochastic Volatility

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1. Introduction

An option is a contract that gives the holder the right to buy, in the case of a call, or sell, in the case of a put, an underlying asset at a pre-determined strike price. A European option allows the holder to exercise the option only on a pre-determined expiration date, while with an American option the holder can exercise the option at any point in time until the maturity date. Options can incorporate dividends, which are a portion of a company's earning distributed to its shareholders, that can be issued as cash payments, as shares of stock or other property.

Black and Scholes (1973) derived a closed form solution for the value of European options with constant volatility, while Heston (1993) provides a solution for European options with stochastic volatility. It was proved that assuming constant volatility leads to considerable mispricing. Bakshi, Cao and Chen (1997) did a series of tests comparing the Black and Scholes (1973) with three models which allow for stochastic volatility. They showed that incorporating stochastic volatility reduces the absolute pricing error by 20% to 70%. For example a call option with the price \$1.68, under the Black and Scholes model has an error of \$0.78, while with a model with stochastic volatility the error is reduced to \$0.42. Hence, models that allow the volatility of the underlying asset to be stochastic better describe the market behaviour.

Unlike European options, American options do not have a closed form solution for its value with constant or stochastic volatility, due to the fact that the price depends on the optimal exercise policy. The models on American options under stochastic volatility can be separted in two approaches: the Partial Differential Equation, PDE, based and the non PDE based.

There are various numerical methods to price American options. For example, Brennan and Schwartz (1977) introduced finite difference methods; the least squares Monte Carlo is a model developed by Longstaff and Schwartz (2001), where the model uses simulations of cash flows generated by the option and compare them to the value of immediate exercise to calculate the price. In Beliaeva and Nawalkha (2010) a bivariate tree is used where two independent trees are created for the stock price and for the variance. Broadie and Detemple (1996) developed a method for lower and upper bounds on the prices of American options based on regression coefficients. In the Clarke and Parrott (1999) model they use the Heston PDE, transformed into a non dimensional form, with a multigrid iteration method to solve the problem of option pricing. Detemple and Tian (2002) determine the exercise region by a single exercise boundary under general conditions on the interest rate and the dividend yield and derive a recursive integral equation for the exercise boundary.

In this work, we will develop an implementation based on the Heston model with the explicit method. First, we will derive the Heston PDE, showing how it is used in the method described. Then we will test the accuracy of the results, randomly creating options and using the various methods to price them and calculate the errors of each method.

2. Heston Model

2.1. Processes for the stock price and variance

The Heston model assumes that the stock price, S_t, follows a stochastic process

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_1(t) \tag{2.1}$$

and the variance, v_t, folllows a Cox, Ingersoll, and Ross (1985) process

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} S_t dW_2(t), \tag{2.2}$$

where $E^{\mathbb{Q}}[dW_1(t)dW_2(t)|\mathcal{F}_t] = \rho dt$.

The processes in equations (2.1) and (2.2) are defined under the physical measure $\mathbb{P}$.

The parameters of the model are

(1) μ the drift of the process for the stock price;
(2) κ the mean reversion speed for the variance;
(3) θ the mean reversion level for the variance;
(4) σ the volatility of the variance;
(5) v_0 the initial level of the variance;
(6) ρ the correlation between the two Brownian motions $W_1(t)$ and $W_2(t)$;
(7) λ the volatility risk parameter.

The volatility $\sqrt{v_t}$ is modeled through the variance v_t. The process for the variance implies the Ornstein-Uhlenbeck process for the volatility where $h_t = \sqrt{v_t}$ is given by

$$dh_t = -\beta h_t dt + \delta dW_2(t). \tag{2.3}$$

Applying Ito's lemma to equation (2.3), with $v_t = h_t^2$ and $f(h_t)=h_t^2$ we obtain

$$\begin{aligned} dh_t^2 &= f'(h_t)dh_t + 0.5f''(h_t)\delta^2 dt \\ &= 2h_t(-\beta h_t dt + \delta dW_2(t)) + 2 \times 0.5\delta^2 dt \\ &= -2\beta h_t^2 dt + \delta^2 dt + 2h_t \delta dW_2(t) \\ &= (\delta^2 - 2\beta v_t)dt + 2\delta\sqrt{v_t} dW_2(t). \end{aligned} \tag{2.4}$$

Defining $\kappa = 2\beta$, $\theta = \delta^2/(2\beta)$, and $\sigma = 2\delta$, transforms the equation (2.4) into (2.2).

For the princing of options, we need S_t and v_t under the risk neutral measure $\mathbb{Q}$.

The risk neutral process for the stock price is

$$dS_t = rS_t dt + \sqrt{v_t} S_t d\tilde{W}_1(t) \tag{2.5}$$

where

$$\tilde{W}_1(t) = \left(W_1(t) + \frac{\mu - r}{\sqrt{v_t}}\ t\right). \tag{2.6}$$

For the variance, the process is

$$dv_t = [\kappa(\theta - v_t) - \lambda(S_t, v_t, t)]dt + \sigma\sqrt{v_t} d\tilde{W}_2(t) \tag{2.7}$$

where

$$\tilde{W}_2(t) = \left(W_2(t) + \frac{\lambda(S_t, v_t, t)}{\sigma\sqrt{v_t}}\ t\right). \tag{2.8}$$

The function $\lambda(S_t, v_t, t)$ represents the volatility risk premium and is equal to λv_t, where λ is a constant.

Substituting for λv_t in equation (2.7), the variance process under the risk neutral measure is

$$dv_t = \kappa^*(\theta^* - v_t)dt + \sigma\sqrt{v_t} d\tilde{W}_2(t) \tag{2.9}$$

where $\kappa^* = \kappa + \lambda$ and $\theta^* = \kappa\theta/(\kappa+\lambda)$.

Equations (2.5) and (2.9) define the risk neutral process.

2.2. Heston PDE

To derive the Heston PDE we need to form a portfolio consisting of one option $V{=}V(S, v, t)$, Δ units of the stock, and φ units of another option $U(S, v, t)$ for the volatility hedge. The portfolio has value

$$\Pi = V + \Delta S + \varphi U. \tag{2.10}$$

Assuming that the portfolio is self financing, the change in portfolio value is

$$d\Pi = dV + \Delta dS + \varphi dU. \tag{2.11}$$

We apply Ito's lemma to the value of $dV(s, v, t)$, and using the fact that

$$\begin{aligned}
&(dS)^2 = vS^2(dW_1(t))^2{=}vS^2dt \\
&(dv)^2 = \sigma^2 v dt \\
&dSdv = \sigma v S dW_1(t)dW_2(t) = \sigma\rho v S dt \\
&(dt)^2 = 0 \\
&dW_1(t)dt = dW_2(t)dt = 0,
\end{aligned} \tag{2.12}$$

and we get the following equation

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial v}dv + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2}dt + \frac{1}{2}v\sigma^2\frac{\partial^2 V}{\partial v^2}dt + \sigma\rho vS\frac{\partial^2 V}{\partial S\partial v}dt. \tag{2.13}$$

Applying Ito's lemma to $dU(S, v, t)$, we obtain a similiar equation to (2.13) but in terms of U.

Substituting the expressions of $dV(S, v, t)$ and $dU(S, v, t)$ into equation (2.11), the change in portfolio value can be written as

$$\begin{aligned} d\Pi &= dV + \Delta dS + \varphi dU \\ &= \left[\frac{\partial V}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \sigma\rho vS\frac{\partial^2 V}{\partial S\partial v} + \frac{1}{2}v\sigma^2\frac{\partial^2 V}{\partial v^2}\right]dt \\ &+ \varphi\left[\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \sigma\rho vS\frac{\partial^2 U}{\partial S\partial v} + \frac{1}{2}v\sigma^2\frac{\partial^2 U}{\partial v^2}\right]dt \\ &+ \left[\frac{\partial V}{\partial S} + \varphi\frac{\partial U}{\partial S} + \Delta\right]dS + \left[\frac{\partial V}{\partial v} + \varphi\frac{\partial U}{\partial v}\right]dv. \end{aligned} \tag{2.14}$$

In order for the portfolio to be hedged against movements in both the stock and volatility, the last two terms in the last equation must be zero. This implies that

$$\varphi = -\frac{\partial V}{\partial v}\bigg/\frac{\partial U}{\partial v} \tag{2.15}$$

and

$$\Delta = -\varphi\frac{\partial U}{\partial S} - \frac{\partial V}{\partial S}. \tag{2.16}$$

Substituting these values in equation (2.14) we obtain

$$\begin{aligned} d\Pi = &\left[\frac{\partial V}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \sigma\rho vS\frac{\partial^2 V}{\partial S\partial v} + \frac{1}{2}v\sigma^2\frac{\partial^2 V}{\partial v^2}\right]dt \\ + \varphi&\left[\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \sigma\rho vS\frac{\partial^2 U}{\partial S\partial v} + \frac{1}{2}v\sigma^2\frac{\partial^2 U}{\partial v^2}\right]dt. \end{aligned} \tag{2.17}$$

The condition that the portfolio earn the risk free rate, r, implies that the change in portfolio value is $d\Pi{=}r\Pi dt$, transforming Equation (2.11) into

$$d\Pi = r(V + \Delta S + \varphi U)dt. \tag{2.18}$$

Combining equations (2.17) and (2.18), and using (2.15) and (2.16), we obtain

$$
\begin{aligned}
&\frac{\left[\frac{\partial V}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \sigma\rho vS\frac{\partial^2 V}{\partial S\partial v} + \frac{1}{2}v\sigma^2\frac{\partial^2 V}{\partial v^2}\right] - rV + rS\frac{\partial V}{\partial S}}{\frac{\partial V}{\partial v}} \\
&= \frac{\left[\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \sigma\rho vS\frac{\partial^2 U}{\partial S\partial v} + \frac{1}{2}v\sigma^2\frac{\partial^2 U}{\partial v^2}\right] - rU + rS\frac{\partial U}{\partial S}}{\frac{\partial U}{\partial v}}.
\end{aligned} \tag{2.19}
$$

As both sides of the equation are expressed only in terms of V and U, respectively, they can be written as a function $f(S, v, t)$. Heston specifies this function as

$$
f(S, v, t) = -\kappa(\theta - v) + \lambda(S, v, t). \tag{2.20}
$$

Substituting the left hand side of equation (2.19) with the funtion $f(S, v, t)$ we obtain

$$
\begin{aligned}
&-\kappa(\theta - v) + \lambda(S, v, t) = \\
&= \frac{\left[\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \sigma\rho vS\frac{\partial^2 U}{\partial S\partial v} + \frac{1}{2}v\sigma^2\frac{\partial^2 U}{\partial v^2}\right] - rU + rS\frac{\partial U}{\partial S}}{\frac{\partial U}{\partial v}}.
\end{aligned} \tag{2.21}
$$

Rearranging the previous equation, we produce the Heston PDE expressed in terms of the price S

$$
\begin{aligned}
&\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \sigma\rho vS\frac{\partial^2 U}{\partial S\partial v} + \frac{1}{2}v\sigma^2\frac{\partial^2 U}{\partial v^2} \\
&- rU + rS\frac{\partial U}{\partial S} + [\kappa(\theta - v) - \lambda(S, v, t)]\frac{\partial U}{\partial v} = 0.
\end{aligned} \tag{2.22}
$$

Defining $x = \ln S$, we can express the PDE in terms of (x, v, t) instead of (S, v, t), using the follow derivatives

$$
\frac{\partial U}{\partial S} = \frac{\partial U}{\partial x}\frac{1}{S} \tag{2.23}
$$

$$
\frac{\partial^2 U}{\partial v\partial S} = \frac{\partial}{\partial v}\left(\frac{1}{S}\frac{\partial U}{\partial x}\right) = \frac{1}{S}\frac{\partial^2 U}{\partial v\partial x} \tag{2.24}
$$

and

$$
\frac{\partial^2 U}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial U}{\partial x}\right) = -\frac{1}{S^2}\frac{\partial U}{\partial x} + \frac{1}{S}\frac{\partial^2 U}{\partial S\partial x} = -\frac{1}{S^2}\frac{\partial U}{\partial x} + \frac{1}{S^2}\frac{\partial^2 U}{\partial x}. \tag{2.25}
$$

Substituting in equation (2.22), we obtain the Heston PDE in terms of $x = \ln S$

$$
\begin{aligned}
&\frac{\partial U}{\partial t} + \frac{1}{2}v\frac{\partial^2 U}{\partial x^2} + \sigma\rho v\frac{\partial^2 U}{\partial v\partial x} + \frac{1}{2}v\sigma^2\frac{\partial^2 U}{\partial v^2} \\
&- rU + \left(r - \frac{1}{2}v\right)\frac{\partial U}{\partial x} + [\kappa(\theta - v) - \lambda v]\frac{\partial U}{\partial v} = 0.
\end{aligned} \tag{2.26}
$$

2.3. Dividends

The Heston PDE can be written to include dividends into the model. Assuming that the dividend payment is a continuous yield, q, we re-write equation (2.5) replacing r by $r - q$

$$dS_t = (r - q)S_t dt + \sqrt{v_t} S_t d\tilde{W}_1. \tag{2.27}$$

Following the process described for the Heston PDE without dividends, we obtain a variation of equation (2.26)

$$\begin{aligned} &\frac{\partial U}{\partial t} + \frac{1}{2} v \frac{\partial^2 U}{\partial x^2} + \sigma \rho v \frac{\partial^2 U}{\partial v \partial x} + \frac{1}{2} v \sigma^2 \frac{\partial^2 U}{\partial v^2} \\ &- rU + \left(r - q - \frac{1}{2} v\right) \frac{\partial U}{\partial x} + [\kappa(\theta - v) - \lambda v] \frac{\partial U}{\partial v} = 0. \end{aligned} \tag{2.28}$$

With dividends, the price of a European call and put are, respectively

$$C(K) = e^x e^{-q\tau} P_1 - K e^{-r\tau} P_2 \tag{2.29}$$

and

$$P(K) = e^x e^{-q\tau}(1 - P_1) + K e^{-r\tau}(1 - P_2), \tag{2.30}$$

where $P_1 = \mathbb{Q}^S(S_T > K)$ and $P_2 = \mathbb{Q}(S_T > K)$ are the in the money probabilities.

The price of an option is represented as a function of the underlying asset price S, the volatility v and the time τ, $U(S, v, \tau)$.

The price of American options satisfies the Heston PDE

$$\begin{aligned}\frac{\partial U}{\partial t} &+ \frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \sigma \rho v S \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2} v \sigma^2 \frac{\partial^2 U}{\partial v^2} \\ &- rU + (r - q) S \frac{\partial U}{\partial S} + [\kappa(\theta - v) - \lambda(S, v, t)] \frac{\partial U}{\partial v} = 0.\end{aligned} \tag{3.1}$$

The term $U^n_{i,j} = U(S_i, v_j, t_n)$ represents the value of the derivative when the stock price, volatility and maturity are at points i, j and n respectively of their grids, for $i = 0,...,N_S$, for $j = 0,...,N_V$ and for $n = 0,...,N_T$.

To solve the PDE we need the following boundary conditions

$$\begin{aligned}&U^0_{i,j} = (K - S_i)^+ \\ &U^{n+1}_{i,j} = max(K - S_i, U^{n+1}_{i,j}) \\ &U^n_{N_S,j} = max(K - S_{N_S}, 0) \\ &U^n_{i,N_V} = max(K - S_i, 0).\end{aligned} \tag{3.2}$$

We defined the finite difference approximations as

$$\begin{aligned}&\frac{\partial U}{\partial S} = (U^n_{i+1,j} - U^n_{i-1,j})/2\delta_S \\ &\frac{\partial U}{\partial v} = (U^n_{i,j+1} - U^n_{i,j-1})/2\delta_v \\ &\frac{\partial^2 U}{\partial S^2} = (U^n_{i+1,j} - 2U^n_{i,j} + U^n_{i-1,j})/\delta_S^2 \\ &\frac{\partial^2 U}{\partial v^2} = (U^n_{i,j+1} - 2U^n_{i,j} + U^n_{i,j-1})/\delta_v^2 \\ &\frac{\partial^2 U}{\partial S \partial v} = (U^n_{i+1,j+1} - U^n_{i-1,j+1} - U^n_{i+1,j-1} + U^n_{i-1,j-1})/4\delta_S \delta_v \\ &\frac{\partial U}{\partial t} = (U^{n+1}_{i+1,j} - U^n_{i-1,j})/\delta_t\end{aligned} \tag{3.3}$$

where δ_S, δ_v and δ_t represent the difference between two points in the stock price, volatility and maturity grids, respectively.

Substituting the finite difference approximations in equation (3.1), we obtain

$$
\begin{aligned}
\frac{(U_{i,j}^{n+1} - U_{i,j}^{n})}{\delta_t} &= \frac{1}{2} v S^2 \frac{(U_{i+1,j}^{n} - 2U_{i,j}^{n} + U_{i-1,j}^{n})}{\delta_S^2} \\
+\sigma\rho v S &\frac{(U_{i+1,j+1}^{n} - U_{i-1,j+1}^{n} - U_{i+1,j-1}^{n} + U_{i+1,j+1}^{n})}{4\delta_S\delta_v} \\
&+ \frac{1}{2} v \sigma^2 \frac{(U_{i,j+1}^{n} - 2U_{i,j}^{n} + U_{i,j-1}^{n})}{\delta_v^2} \\
&-rU_{i,j}^{n} \\
&+(r-q)S\frac{(U_{i+1,j}^{n} - U_{i-1,j}^{n})}{2\delta_S} \\
&+\kappa(\theta - v)\frac{(U_{i,j+1}^{n} - U_{i,j-1}^{n})}{2\delta_v}.
\end{aligned} \tag{3.4}
$$

Joining the same index terms

$$
\begin{aligned}
\frac{(U_{i,j}^{n+1} - U_{i,j}^{n})}{\delta_t} &= U_{i+1,j}^{n}\left(\frac{1}{2\delta_S^2} v S^2 + \frac{(r-q)S}{2\delta_S}\right) + \\
&U_{i-1,j}^{n}\left(\frac{1}{2\delta_S^2} v S^2 - \frac{(r-q)S}{2\delta_S}\right) + \\
&U_{i,j+1}^{n}\left(\frac{1}{2\delta_v^2} v \sigma^2 + \frac{\kappa(\theta - v)}{2\delta_v}\right) + \\
&U_{i,j-1}^{n}\left(\frac{1}{2\delta_v^2} v \sigma^2 - \frac{\kappa(\theta - v)}{2\delta_v}\right) + \\
\frac{\sigma\rho v S}{4\delta_S\delta_v}&\left(U_{i+1,j+1}^{n} - U_{i-1,j+1}^{n} - U_{i+1,j-1}^{n} + U_{i+1,j+1}^{n}\right) + \\
&U_{i,j}^{n}\left(-\frac{vS^2}{\delta_S^2} - \frac{\sigma^2 v}{\delta_v^2} - r\right).
\end{aligned} \tag{3.5}
$$

To solve this PDE, firstly, we need to create grids for the stock price, volatility and maturity. Then we need to choose a finite difference methodology to solve the PDE.

We will use the explicit method, which defines the value of the derivate at maturity point $n+1$ as

$$
\begin{aligned}
U_{i,j}^{n+1} = U_{i,j}^{n} + dt\Bigg[&\frac{1}{2} v_j S_i^2 \frac{\partial^2}{\partial S^2} + \sigma\rho v_j S_i \frac{\partial^2}{\partial S \partial v} \\
&+ \frac{1}{2} v_j \sigma^2 \frac{\partial^2}{\partial v^2} - r + (r-q) S_i \frac{\partial}{\partial S} + \kappa(\theta - v)\frac{\partial}{\partial v}\Bigg] U_{i,j}^{n}.
\end{aligned} \tag{3.6}
$$

Transforming equation (3.5) into the form of equation (3.6), we obtain

$$
\begin{aligned}
U_{i,j}^{n+1} = U_{i,j}^{n} + \Bigg[& U_{i+1,j}^{n}\left(\frac{1}{2\delta_S^2}vS^2 + \frac{(r-q)S}{2\delta_S}\right) + \\
& U_{i-1,j}^{n}\left(\frac{1}{2\delta_S^2}vS^2 - \frac{(r-q)S}{2\delta_S}\right) + \\
& U_{i,j+1}^{n}\left(\frac{1}{2\delta_v^2}v\sigma^2 + \frac{\kappa(\theta-v)}{2\delta_v}\right) + \\
& U_{i,j-1}^{n}\left(\frac{1}{2\delta_v^2}v\sigma^2 - \frac{\kappa(\theta-v)}{2\delta_v}\right) + \\
\frac{\sigma\rho vS}{4\delta_S\delta_v} & \left(U_{i+1,j+1}^{n} - U_{i-1,j+1}^{n} - U_{i+1,j-1}^{n} + U_{i+1,j+1}^{n}\right) + \\
& U_{i,j}^{n}\left(-\frac{vS^2}{\delta_S^2} - \frac{\sigma^2 v}{\delta_v^2} - r\right)\Bigg]\delta_t.
\end{aligned}
\tag{3.7}
$$

This is the equation used in the implementation of the method in Matlab with the boundary conditions in equation (3.2)

$$
\begin{aligned}
U_{i,j}^{n+1} &= U_{i,j}^{n} + [U_{i+1,j}^{n}D_1 + U_{i-1,j}^{n}D_2 + U_{i,j+1}^{n}D_3 + U_{i,j-1}^{n}D_4 \\
&+ (U_{i+1,j+1}^{n} - U_{i-1,j+1}^{n} - U_{i+1,j-1}^{n} + U_{i+1,j+1}^{n})D_5 + U_{i,j}^{n}D_6]\delta_t,
\end{aligned}
\tag{3.8}
$$

where

$$
\begin{aligned}
D_1 &= \frac{1}{2\delta_S^2}vS^2 + \frac{(r-q)S}{2\delta_S} \\
D_2 &= \frac{1}{2\delta_S^2}vS^2 - \frac{(r-q)S}{2\delta_S} \\
D_3 &= \frac{1}{2\delta_v^2}v\sigma^2 + \frac{\kappa(\theta-v)}{2\delta_v} \\
D_4 &= \frac{1}{2\delta_v^2}v\sigma^2 - \frac{\kappa(\theta-v)}{2\delta_v} \\
D_5 &= \frac{\sigma\rho vS}{4\delta_S\delta_v} \\
& \mathit{and} \\
D_6 &= -\frac{vS^2}{\delta_S^2} - \frac{\sigma^2 v}{\delta_v^2} - r.
\end{aligned}
\tag{3.9}
$$

4. Model comparisons

In this section, we are going to describe various methods that are used in the pricing of American options under the Heston model. We are going to compare the results of the methods, which are obtained with the codes of Rouah (2013) book, with the results of the method in Section 3, to see if these results are an improvement regarding the other models. We will use the Least Squares Monte Carlo model as a benchmark to test the results.

4.1. Least-Squares Monte Carlo

This method was developed by Longstaff and Schwartz (2001), using simulations to price American options. The algorithm is based on the function $C(\omega, s; t, T)$ that denotes the set of cash flows generated by the option along the stock price path ω, with the condition that the option is not exercised prior to time t, and the holder follows the optimal stopping strategy at all times.

The value of continuing to hold the option, $F(\omega, t_k)$, at time t_k, is defined as

$$F(\omega, t_k) = e^{-r(T-t_k)} E^{\mathbb{Q}} \left[\sum_{j=k+1}^{K} C(\omega, t_j; t_k, T) | \mathcal{F}_{t_k} \right] \tag{4.1}$$

assuming a constant rate of interest r and using the risk-neutral measure $\mathbb{Q}$.

To evaluate the option we need to compare the value of immediate exercise with $F(\omega, t_k)$, which needs to be estimated because it is unknown.

Longstaff and Schwartz (2001) estimate $F(\omega, t_k)$ using least squares on a set of basis functions, which they select to be the weighted Laguerre polynomials and a basis of $L^2([0, +\infty[)$

$$\begin{aligned} L_0(x) &= e^{-x/2} \\ L_1(x) &= e^{-x/2}(1-x) \\ L_2(x) &= e^{-x/2}(1-2x+x^2/2) \\ L_M(x) &= e^{-x/2} \sum_{j=0}^{M} \frac{(-1)^r}{r!} \binom{M}{r} x^r. \end{aligned} \tag{4.2}$$

$F(\omega, t_k)$ can be approximated, using the first M basis functions by

$$F(\omega, t_k) = \sum_{j=0}^{M} a_j L_j(S_k), \tag{4.3}$$

where $S_k = S_k(\omega)$ is the value of the underlying stock price at time t_k along the price path ω. The coefficients a_j are constants that are estimated using least squares.

Equation (4.3) can be rewritten in matrix form, **F**=**La**, where

$$\mathbf{F}_{K\times 1} = \begin{bmatrix} F_M(\omega, t_1) \\ F_M(\omega, t_2) \\ \vdots \\ F_M(\omega, t_K) \end{bmatrix}$$

$$\mathbf{L}_K = \begin{bmatrix} L_0(S_1) & L_1(S_1) & \dots & L_M(S_1) \\ L_0(S_2) & L_1(S_2) & \dots & L_M(S_2) \\ \vdots & \vdots & \ddots & \vdots \\ L_0(S_K) & L_1(S_K) & \dots & L_M(S_K) \end{bmatrix}$$

$$\mathbf{a}_{N\times 1} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_M \end{bmatrix}.$$

The cash flows at time t_k depend on whether or not exercise occurs at t_{k+1}. So they have to be determined starting at t_{K-1} until the moment t_2. At t_K=T the cash flow is the payoff. For t_k, with $2 \leq k \leq K-1$, the stock price paths in the money are choosen and is calculated the immediate exercise value for those paths.

They estimate the $M+1$ coefficients a_0,...,a_M of equation (4.3) by regression, using the basis functions in a design matrix and using the single-period discounted cash flows as the dependent variable. The least squares regression estimates are

$$\hat{\mathbf{a}} = (\mathbf{L}'\mathbf{L})^{-1}\mathbf{L}'\mathbf{F}. \tag{4.4}$$

Then the predicted continuation value, i.e., the predicted cash flow, is calculated

$$\hat{F}(\omega, t_{k-1}) = \hat{a}_0 L_0(S_{k-1}) + \hat{a}_1 L_1(S_{k-1}) + ... + \hat{a}_M L_M(S_{k-1}), \tag{4.5}$$

which is compared to the value of immediate exercise. At the paths that the value of immediate exercise is greater than the predicted cash flows, the value of the cash flows is updated with the value of immediate exercise.

The value of the option is then the average of the new cash flows of all paths updated to time t_1.

4.2. Beliaeva-Nawalkha Bivariate Tree

The concept of this method is to create separated and independent trees for the stock price and for the variance, and then combining the two trees. To have independent trees, the process S_t needs to be transform into Y_t, that is independet of v_t.

The Heston model is defined by these two equations, as explained in section 2

$$\begin{aligned} dS_t &= (r-q)S_t dt + \sqrt{v_t} S_t dW_1(t) \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} S_t dW_2(t) \end{aligned} \tag{4.6}$$

where $E^{\mathbb{Q}}[dW_1(t)dW_2(t)|\mathcal{F}_t] = \rho dt$.

The process Y_t is chosen to be defined by

$$Y_t = lnS_t - \frac{\rho}{\sigma}v_t - h_t \tag{4.7}$$

where

$$h_t = \Big(r - \frac{\rho\kappa\theta}{\sigma}\Big)t. \tag{4.8}$$

Applying Ito's lemma produces the equation

$$dY_t = \mu_Y(t)dt + \sigma_Y(t)\sqrt{v_t}dW_1(t)^* \tag{4.9}$$

where

$$\begin{aligned} \mu_Y(t) &= \Big(\frac{\rho\kappa}{\sigma} - \frac{1}{2}\Big)v_t \\ \sigma_Y(t) &= \sqrt{1-\rho^2}\sqrt{v_t} \end{aligned} \tag{4.10}$$

with

$$dW_1(t)^* = \frac{dW_1(t) - \rho dW_2(t)}{\sqrt{1-\rho^2}}. \tag{4.11}$$

Since $E^{\mathbb{Q}}[dW_1(t)^* dW_2(t)|\mathcal{F}_t] = 0$, the processes Y_t and v_t are independent, and can be approximated with trinomial trees. Which means that the joint probabilities in the tree (Y_t,v_t) will be the product of the marginal probabilities for Y_t and v_t.

Trinomial Tree for the Variance

Beliaeva and Nawalkha (2010) build a trinomial tree for the transformed variance x_t defined as

$$x_t = \frac{2\sqrt{v_t}}{\sigma}. \tag{4.12}$$

They recover the variance v_t through the inverse transformation

$$v_t = \frac{1}{4}x_t^2\sigma^2. \tag{4.13}$$

A trinomial tree for x_t is constructed first, and transformed into a tree for the variances v_t through equation (4.13).

By Ito's lemma x_t follows a SDE with drift

$$\mu(x_t, t) = \frac{1}{x_t}\left(\frac{2\kappa\theta}{\sigma^2} - \frac{\kappa x_t^2}{2} - \frac{1}{2}\right). \tag{4.14}$$

The time zero node of the trinomial tree for x_t is x_0, and is obtained by substituting v_0 into equation (4.12). At time $t>0$, given that the process is at node x_t, there are two sets of moves.

Case 1 : If $x_t>0$, the up, middle and down moves at time $t+dt$ are

$$\begin{aligned} x^u_{t+dt} &= x_t + b(J+1)\sqrt{dt} \\ x^m_{t+dt} &= x_t + bJ\sqrt{dt} \\ x^d_{t+dt} &= x_t + b(J-1)\sqrt{dt} \end{aligned} \tag{4.15}$$

where J and b are defined by

$$J = \lfloor\left(\frac{\mu(x_t,t)\sqrt{dt}}{b} + \frac{1}{b^2}\right)\rfloor \tag{4.16}$$

and

$$b = \begin{cases} b_c, & \text{if } |b_c - \sqrt{1.5}| < |b_e - \sqrt{1.5}| \\ b_e, & \text{otherwise} \end{cases} \tag{4.17}$$

with

$$b_e = \frac{x_0/\sqrt{dt}}{\lfloor(x_0/\sqrt{1.5dt})\rfloor} \tag{4.18}$$

and

$$b_c = \frac{x_0/\sqrt{dt}}{\lfloor(x_0/\sqrt{1.5dt}+1)\rfloor}. \tag{4.19}$$

The probability of each move is

$$
p_v^u = \frac{1}{2b^2} - \frac{J}{2} + \frac{1}{2b}\mu(x_t, t)\sqrt{dt}
$$
$$
p_v^m = 1 - \frac{1}{b^2} \tag{4.20}
$$
$$
p_v^d = \frac{1}{2b^2} + \frac{J}{2} - \frac{1}{2b}\mu(x_t, t)\sqrt{dt}
$$

Case 2 : If x_t=0, the up move x_{t+dt}^u is defined identically to that in equation (4.15), the down move is x_{t+dt}^d=0, and the middle move x_{t+dt}^m does not exist. The probabilities in this case are

$$
p_v^u = \frac{\kappa\theta dt}{v_{t+dt}^u}
$$
$$
p_v^m = 0 \tag{4.21}
$$
$$
p_v^d = 1 - p_v^u
$$

where v_{t+dt}^u is obtained by substituting x_{t+dt}^u into equation (4.13).

The b parameter is defined within the range $1 \leq b \leq \sqrt{2}$ and serves to contract or expand the tree to ensure that the last row of the tree for x_t is exactly zero. The trinomial tree for v_t is obtained by substituting the value x_t at each node into equation (4.13).

Trinomial Tree for the Stock Price

Given a value Y_t, the stock price can be recovered by inverting the equation (4.7)

$$
S_t = exp\left(Y_t + \frac{\rho}{\sigma}v_t + h_t\right). \tag{4.22}
$$

High values of v_t cause Y_t to jump up and down across multiple nodes while low values of v_t allow jumps across single nodes only. Beliaeva and Nawalkha (2010) define the node span as $k_t\sigma_Y(0)\sqrt{dt}$, which represents the distance between nodes for values of Y_{t+dt}, given that the process is at the node Y_t.

The case k_t=1 represents a jump across a single node, while k_t>1 represents a jump across multiple nodes. This parameter is defined as

$$k_t = \begin{cases} \lceil(\sqrt{v_t/v_0})\rceil, & \text{if } v_t > 0 \\ 1, & \text{otherwise} \end{cases} . \tag{4.23}$$

The initial node of the tree at time zero is given by Y_0, obtained by setting t=0 in equations (4.7) and (4.8).

The up, middle, and down values of Y_{t+dt} are

$$\begin{aligned} Y^u_{t+dt} &= Y_t + (I+1)k_t\sigma_Y(0)\sqrt{dt} \\ Y^m_{t+dt} &= Y_t + Ik_t\sigma_Y(0)\sqrt{dt} \\ Y^d_{t+dt} &= Y_t + (I-1)k_t\sigma_Y(0)\sqrt{dt} \end{aligned} \tag{4.24}$$

where I is the integer closest in absolute value to

$$\frac{\sigma_Y(t)\sqrt{dt}}{k_t\sigma_Y(0)}. \tag{4.25}$$

The probabilities of up, middle, and down moves are given by

$$\begin{aligned} p^u_Y &= \frac{1}{2}\frac{\sigma_Y(t)^2dt + e_me_d}{(k_t\sigma_Y(0))^2dt} \\ p^m_Y &= -\frac{\sigma_Y(t)^2dt + e_ue_d}{(k_t\sigma_Y(0))^2dt} \\ p^d_Y &= \frac{1}{2}\frac{\sigma_Y(t)^2dt + e_ue_m}{(k_t\sigma_Y(0))^2dt} \end{aligned} \tag{4.26}$$

where

$$\begin{aligned} e_u &= Y^u_{t+dt} - Y_t - \mu_Y(t)dt = (I+1)k_t\sigma_Y(0)\sqrt{dt} - \mu_Y(t)dt \\ e_m &= Y^m_{t+dt} - Y_t - \mu_Y(t)dt = Ik_t\sigma_Y(0)\sqrt{dt} - \mu_Y(t)dt \\ e_d &= Y^d_{t+dt} - Y_t - \mu_Y(t)dt = (I-1)k_t\sigma_Y(0)\sqrt{dt} - \mu_Y(t)dt. \end{aligned} \tag{4.27}$$

The tree for the stock price S_t is obtained by applying the inverse transformation of (4.22) at every node of the tree for Y_t.

Combining the Trinomial Trees

The final step is to merge the trinomial trees of v_t and S_t into a single tree. At time zero there is a single node for (S_0, v_0). At each node (S_t, v_t) of the tree, S_t and v_t have three possible values, up, middle, or down, respectively. Hence, each node (S_0, v_0) produces 3×3=9 potencial new nodes.

Since these nodes recombine, however, the actual number of nodes does not increase by a factor of nine at each time step. Rather the number of nodes depends on the values of k_t at the nodes. The number of nodes can increase very rapidly but the fact that the tree for Y_t recombines mitigates this increase substantially.

Since the trees for Y_t and v_t are uncorrelated, the joint probabilities of these branches are the product of the three marginal probabilities from each tree, defined in equations (4.20) and (4.26)

$$\begin{array}{lllllllll} p_{uu} & = & p_Y^u \times p_v^u & p_{mu} & = & p_Y^m \times p_v^u & p_{du} & = & p_Y^d \times p_v^u \\ p_{um} & = & p_Y^u \times p_v^m & p_{mm} & = & p_Y^m \times p_v^m & p_{dm} & = & p_Y^d \times p_v^m \\ p_{ud} & = & p_Y^u \times p_v^d & p_{md} & = & p_Y^m \times p_v^d & p_{dd} & = & p_Y^d \times p_v^d. \end{array} \tag{4.28}$$

With the tree for the stock price S_t and the joint probabilities, pricing american options is done exactly as in an ordinary trinomial tree, by working backward in time from the maturity where the payoff is known, and at each node comparing the value of the american option with the value of immediate exercise.

The price of the american put at time t is

$$\begin{aligned} U(S_t, v_t) = e^{-r \times dt} max(K - S_t, p_{uu} U(S^u_{t+dt}, v^u_{t+dt}) \\ + p_{um} U(S^u_{t+dt}, v^m_{t+dt}) + ... + p_{dd} U(S^d_{t+dt}, v^d_{t+dt})). \end{aligned} \tag{4.29}$$

4.3. Medvedev-Scaillet Expansion

The approximation for the American put price under the Heston model for the Medvedev-Scaillet expansion is

$$P(\theta, \tau, v) = \sum_{n=1}^{\infty} P_n(\theta, v) \tau^{n/2} \tag{4.30}$$

where

$$\theta = \frac{ln(K/S)}{\sqrt{v}\sqrt{\tau}} \tag{4.31}$$

and

$$\begin{aligned} P_n(\theta, v) = C_n(v)[p_n^0(\theta)\Phi(\theta) + q_n^0(\theta)\phi(\theta)] + p_n^1(\theta, v)\Phi(\theta) + q_n^1(\theta, v)\phi(\theta) \\ = C_n(v) P_n^0(\theta) + P_n^1(\theta, v) \end{aligned} \tag{4.32}$$

where $\Phi(\theta)$ and $\phi(\theta)$ denote the standard normal cumulative distribution function and density, respectively.

We re-write the Heston PDE using the notation of Medvedev-Scaillet (2010), using θ' to represent the mean reversion level of the variance process

$$P_t + (r-q)SP_S + \frac{1}{2}vS^2P_{SS} - rP + \rho\sigma vSP_{vS} + \frac{1}{2}\sigma^2 vP_{vv} + \kappa(\theta'-v)\mathbf{P}_v = 0. \tag{4.33}$$

To transform $P(S,v,t)$ into $P(\theta,v,\tau)$, we need the following derivatives

$$P_t = \frac{\theta}{2\tau}P_\theta - P_\tau \tag{4.34}$$

$$P_S = P_\theta\theta_S = \frac{-1}{S\sqrt{v}\sqrt{\tau}}P_\theta \tag{4.35}$$

$$P_{SS} = P_{\theta\theta}(\theta_S)^2 + P_\theta\theta_{SS} = \frac{1}{S^2v\tau}P_{\theta\theta} + \frac{1}{S^2\sqrt{v}\sqrt{\tau}}P_\theta \tag{4.36}$$

$$P_v = P_v + P_\theta\theta_v = P_v - \frac{\theta}{2v}P_\theta \tag{4.37}$$

$$P_{vv} = P_{vv} - \frac{\theta}{v}P_{v\theta} + \frac{\theta^2}{4v^2}P_{\theta\theta} + \frac{3\theta}{4v^2}P_\theta \tag{4.38}$$

(4.39)
$$P_{vS} = P_{v\theta}\theta_S - \frac{1}{2v}(\theta_S P_\theta +_{\theta\theta}\theta_S) = \frac{-1}{S\sqrt{v}\sqrt{\tau}}P_{v\theta} + \frac{1}{2Sv^{3/2}\sqrt{\tau}}P_\theta + \frac{\theta}{2Sv^{3/2}\sqrt{\tau}}P_{\theta\theta}$$

where the derivatives of θ are

$$\theta_S = \frac{-1}{S\sqrt{v}\sqrt{\tau}} \tag{4.40}$$

$$\theta_{SS} = \frac{1}{S^2\sqrt{v}\sqrt{\tau}} \tag{4.41}$$

$$\theta_v = \frac{-\theta}{2v} \tag{4.42}$$

$$\theta_{vv} = \frac{\theta - \theta_v v}{v^2} = \frac{3\theta}{4v^2} \tag{4.43}$$

and

$$\theta_{Sv} = \frac{1}{2Sv^{3/2}\sqrt{\tau}}. \tag{4.44}$$

Substituting the derivatives into the Heston PDE, equation (4.32), and multiplying by 2τ, we obtain

(4.45)
$$\begin{aligned}
&P_{\theta\theta} + \theta P_\theta - 2\tau P_\tau \\
&+ \sqrt{\tau}\left[\frac{1}{\sqrt{\tau}}(v + 2(q-r))P_\theta + \rho\sigma\sqrt{\tau}\left(-2P_{v\theta} + \frac{1}{v}P_\theta + \frac{\theta}{v}P_{\theta\theta}\right)\right] \\
&+ \tau\left[\kappa(\theta' - v)\left(2P_v - \frac{\theta}{v}P_\theta\right) + \sigma^2 v\left(P_{vv} - \frac{\theta}{v}P_{v\theta} + \frac{\theta^2}{4v^2}P_{\theta\theta} + \frac{3\theta}{4v^2}P_\theta\right) - 2rP\right] \\
&= 0.
\end{aligned}$$

We need to express equation (4.45) in terms of $P_n(\theta, v)$. The terms that are multiplied by $\sqrt{\tau}$ get shifted back one in n, and those multiplied by τ get shifted back twice in n, we obtain the equation

$$\begin{aligned}
&P_{n\theta\theta} + \theta P_{n\theta} - 2\tau P_n \\
&+ \frac{1}{\sqrt{\tau}}(v + 2(q-r))P_{n-1,\theta} + \rho\sigma\sqrt{\tau}\left(-2P_{n-1,v\theta} + \frac{1}{v}P_{n-1,\theta} + \frac{\theta}{v}P_{n-1,\theta\theta}\right) \\
&+ \kappa(\theta' - v)\left(2P_{n-2,v} - \frac{\theta}{v}P_{n-2,\theta}\right) \\
&+ \sigma^2 v\left(P_{n-2,vv} - \frac{\theta}{v}P_{n-2,v\theta} + \frac{\theta^2}{4v^2}P_{n-2,\theta\theta} + \frac{3\theta}{4v^2}P_{n-2,\theta}\right) - 2rP_{n-2} = 0.
\end{aligned} \tag{4.46}$$

To find the solution for the re-arranged PDE, we consider the homogeneous and non-homogeneous portions of the PDE separately. The homogeneous part consists of the terms $P_{n\theta\theta} + \theta P_{n\theta} - 2\tau P_n$, and the remaining terms are the non-homogeneous part.

The homogeneous part has the solution

$$P_n^1(\theta, v) = p_n^1(\theta, v)\Phi(\theta) + q_n^1(\theta, v)\phi(\theta) \tag{4.47}$$

while the non-homogeneous part has the following solution

$$\begin{aligned}
P_n(\theta, v) &= C_n(v)P_n^0(\theta) + P_n^1(\theta, v) \\
&= C_n(v)[p_n^0(\theta)\Phi(\theta) + q_n^0(\theta)\phi(\theta)] + p_n^1(\theta, v)\Phi(\theta) + q_n^1(\theta, v)\phi(\theta).
\end{aligned} \tag{4.48}$$

Using the following derivatives of $P_n(\theta, v)$

$$
(4.49)\quad
\begin{aligned}
&P_{n\theta} = C_n[p^0_{n\theta}\Phi + p^0_n\phi + q^0_{n\theta}\phi - q^0_n\theta\phi] + p^1_{n\theta}\Phi + p^1_n\phi + q^1_{n\theta}\phi - q^1_n\theta\phi\\
&P_{n\theta\theta} = C_n[p^0_{n\theta\theta}\Phi + 2p^0_{n\theta}\phi - p^0_n\phi + q^0_{n\theta\theta}\phi - 2q^0_{n\theta}\theta\phi - q^0_n\phi + q^0_n\theta^2\phi]\\
&+ p^1_{n\theta\theta}\Phi + 2p^1_{n\theta}\phi - p^1_n\phi + q^1_{n\theta\theta}\phi - 2q^1_{n\theta}\theta\phi - q^1_n\phi + q^1_n\theta^2\phi\\
&P_{nv} = C_{nv}[p^0_n\Phi + q^0_n\phi] + p^1_{nv}\Phi + q^1_{nv}\phi\\
&P_{nvv} = C_{nvv}[p^0_n\Phi + q^0_n\phi] + p^1_{nvv}\Phi + q^1_{nvv}\phi\\
&P_{nv\theta} = C_{nv}[p^0_{n\theta}\Phi + p^0_n\phi + q^0_n\phi - q^0_n\phi] + p^1_{nv\theta}\Phi + p^1_{nv}\phi + q^1_{nv\theta}\phi - q^1_{nv}\phi,
\end{aligned}
$$

in the solutions of both the homogeneous and non-homogeneous parts, we obtain two equations, one with terms common to $\Phi(\theta)$, and the other with terms common to $\phi(\theta)$.

The homogeneous part of the two equations produces

$$
(4.50)\quad (p^1_{n\theta\theta} + \theta p^1_{n\theta} - np^1_n)\Phi(\theta) + (-(n+1)q^1_n - \theta q^1_{n\theta} + q^1_{n\theta\theta} + 2p^1_{n\theta})\phi(\theta) = 0.
$$

The non homogeneous part, for each n, is solved for p^1_n and q^1_n as all other quantities are know.

The polynomials p^0_n and q^0_n are obtained by recursion. The polynomials p^1_n and q^1_n are expressed as

$$
(4.51)\qquad p^1_n(\theta, v) = \pi^1_{n0}\theta^n + \pi^1_{n1}\theta^{n-2} + \pi^1_{n2}\theta^{n-4} + ...
$$

$$
(4.52)\qquad q^1_n(\theta, v) = x^1_{n1}\theta^{3n-5} + x^1_{n2}\theta^{3n-7} + x^1_{n3}\theta^{3n-9} + x^1_{n4}\theta^{3n-11} + ...
$$

When the polynomials p^0_n, q^0_n, p^1_n and q^1_n are known, we need to constructe the coefficients $C_n(v)$, solving the equation (4.31) with $\theta = y$ and substituting a Taylor series expansion for $exp(\sqrt{vy}\sqrt{y})$. We get the following equation, that can be solved for C_n

$$
(4.53)\quad C_n(v)[p^0_n(y)\Phi_0 + q^0_n(y)\phi_0] + p^1_n(y, v)\Phi_0 + q^1_n(y, v)\phi_0 = \frac{(-1)^{n+1}K}{n!}v^{n/2}y^n.
$$

After finding the polynomials and the coefficients $C_n(v)$, we have to discover the barrier y, defined as

$$
(4.54)\qquad \tilde{y} = arg \max_{y\geq\theta, y\geq 0} P(\theta, \tau, v, y)
$$

where $P(\theta, \tau, v, y)$ is $P(\theta, \tau, v)$ in (4.30), with an extra argument that represents the barrier in $C_n(v)$.

The first approximation of Medvedev and Scaillet (2010) for the price of an american put $P(\theta, \tau, v, \tilde{y})$, given p^0_n, q^0_n, p^1_n, q^1_n and C_n is

(1) Construct $P(\theta, \tau, v, y)$ using equation (4.30) with C_n=$C_n(v, y)$
(2) Find $\tilde{y}$ using equation (4.54), under the constraint $\tilde{y} \geq \theta$
(3) Use C_n=$C_n(v, \tilde{y})$ in $P(\theta, \tau, v, \tilde{y})$ to find the price

4.4. Method Accuracy

To test the accuracy of the method developed in this paper, we chose randomly numbers for the Heston parameters, for the spot, strike, risk free rate, dividend yield and for the maturity to create options. We will use the method of Least Squares Monte Carlo as benchmark.

We used an uniform grid, with the limits $S_{min} = 0$, $S_{max} = 3 \times$ (Strike price), $v_{min} = 0$, $v_{max} = 0.5$, $T_{min} = 0$, $T_{max} =$ Maturity. The number of grid points for the stock, volatility and maturity that we defined to test this method are $nS = 64$, $nv = 34$ and $nT = 5000$ respectively. In the Bivariate Tree method we used 50 time steps.

Firstly we will test the price for American put options with the options created

Table 1. Parameters for American Put Options with S=100

P_t	K	r	q	T	κ	θ	σ	$v0$	ρ	λ
APO1	95	0.05	0.05	0.25	5.82	0.10	0.45	0.33	0.12	0.03
APO2	126	0.00	0.09	0.5	6.60	0.21	0.40	0.38	0.16	0.02
APO3	104	0.08	0.03	0.25	5.75	0.23	0.38	0.14	0.09	0.08
APO4	97	0.02	0.00	0.25	6.26	0.18	0.35	0.09	0.03	0.02
APO5	80	0.01	0.05	0.25	5.09	0.10	0.59	0.07	0.06	0.04
APO6	101	0.02	0.10	0.5	6.37	0.27	0.32	0.24	0.12	0.03
APO7	105	0.05	0.03	1.0	6.89	0.28	0.34	0.10	0.04	0.03
APO8	75	0.05	0.10	0.5	6.34	0.18	0.47	0.12	0.03	0.07
APO9	73	0.04	0.09	0.5	6.67	0.11	0.46	0.35	0.03	0.08
APO10	126	0.05	0.01	0.75	5.03	0.17	0.56	0.18	0.12	0.07
APO11	130	0.10	0.04	0.25	5.06	0.15	0.33	0.37	0.18	0.05
APO12	128	0.02	0.05	0.5	5.54	0.14	0.56	0.32	0.03	0.08
APO13	71	0.00	0.03	0.25	6.94	0.23	0.54	0.33	0.19	0.05
APO14	98	0.07	0.03	0.25	6.12	0.26	0.36	0.38	0.13	0.08
APO15	116	0.01	0.02	0.25	6.71	0.14	0.29	0.09	0.09	0.01
APO16	72	0.01	0.07	0.5	5.98	0.13	0.44	0.32	0.14	0.01
APO17	76	0.03	0.02	1.0	6.09	0.14	0.35	0.32	0.19	0.09
APO18	129	0.06	0.09	0.5	5.44	0.24	0.22	0.22	0.19	0.04
APO19	128	0.09	0.01	0.5	5.13	0.29	0.49	0.12	0.06	0.05
APO20	94	0.02	0.04	0.75	5.90	0.18	0.22	0.13	0.18	0.03

The results of each method for the various American Put Options are presented in the next table, where the column MI is the method implemented.

TABLE 2. Prices of American Put Options

Put Option	L.S. Monte Carlo	MI	BN (2010)	MS (2010)
APO1	6.8093	6.7618	6.1135	6.7301
APO2	35.1756	35.4181	29.8219	34.6415
APO3	10.1427	10.1312	8.9384	10.0612
APO4	5.6204	5.5613	6.7184	5.5607
APO5	0.3671	0.4324	0.3197	0.4004
APO6	16.1800	16.5659	11.0889	16.4014
APO7	21.2704	21.3881	21.0354	20.7283
APO8	2.2695	2.2721	3.7425	2.2373
APO9	2.1490	2.1780	2.6037	2.1642
APO10	30.3461	30.1726	29.0623	30.4680
APO11	31.2393	31.5200	30.9895	31.4784
APO12	32.8614	33.1357	33.4599	32.9610
APO13	1.0425	0.9881	0.5907	1.0007
APO14	9.6962	9.6523	8.5765	9.6905
APO15	17.8580	18.1641	16.5608	18.0845
APO16	2.3167	2.2085	1.5918	2.1685
APO17	5.1168	4.9916	2.9101	5.1023
APO18	34.1687	34.3372	29.4317	34.3056
APO19	30.4527	30.3191	30.4647	29.0240
APO20	11.1397	11.2978	5.3452	11.2620

In this table, we show the various errors relative to the benchmark, where MaxAE, MaXRE, MeanAE, MeanRE and RMSE represent respectively the Maximum Absolute Error, Maximum Relative Error, Mean Absolute Error, Mean Eelative Error and Root Mean Absolute Error.

TABLE 3. Errors for Put Options

Model	MaxAE	MaxRE	MeanAE	MeanRE	RMSE	CPU (s)
MI	0.3859	0.1779	0.1394	0.0063	5.9838	76.076882
Bivariate Tree	5.7945	0.6491	1.7064	0.0914	58.4370	18.735730
M.S. Approximation	1.4287	0.0907	0.2092	0.0043	10.3048	0.201229

We can see that the method implemented takes more time computing than the order models, but has the smallest errors.

The book of Rouah does not have the code to compute the M.S. Approximation method for American call options. Therefore, we will only compare the MI and the Bivariate Tree method with the L.S. Monte Carlo

Table 4. Parameters for American Call Options with S=100

C_t	K	r	q	T	κ	θ	σ	$v0$	ρ	λ
ACO1	73	0.08	0.07	0.25	6.75	0.15	0.29	0.40	0.02	0.02
ACO2	75	0.05	0.02	0.5	5.23	0.14	0.34	0.31	0.18	0.00
ACO3	123	0.06	0.01	0.25	5.44	0.14	0.37	0.16	0.01	0.01
ACO4	114	0.10	0.04	0.25	5.55	0.11	0.51	0.06	0.05	0.04
ACO5	125	0.04	0.02	0.25	5.40	0.22	0.32	0.34	0.19	0.10
ACO6	124	0.02	0.0	0.5	5.10	0.15	0.57	0.22	0.15	0.08
ACO7	125	0.06	0.08	1.0	6.49	0.24	0.36	0.33	0.06	0.0
ACO8	110	0.09	0.03	0.5	5.44	0.29	0.36	0.40	0.06	0.0
ACO9	75	0.046	0.02	0.5	5.86	0.22	0.31	0.11	0.12	0.07
ACO10	72	0.04	0.04	0.75	6.00	0.26	0.33	0.09	0.15	0.04

The results of each model for the various American Call Options are

Table 5. Prices of American Call Options

Call Option	L.S. Monte Carlo	MI	BN (2010)
ACO1	28.055	28.0722	29.7308
ACO2	27.7506	28.2204	50.3495
ACO3	1.7385	1.8309	2.7108
ACO4	1.7836	1.8626	6.4888
ACO5	3.5021	3.5293	6.8429
ACO6	4.8129	4.7988	28.2787
ACO7	10.8726	10.7476	29.6877
ACO8	12.9914	12.5414	23.8747
ACO9	27.9381	28.2819	55.7252
ACO10	31.4076	31.4037	79.0518

Despite having a smaller CPU time, the errors of the Bivariate Tree method are much larger than the method implemented.

Table 6. Errors for Call Options

Model	MaxAE	MaxRE	MeanAE	MeanRE	RMSE	CPU (s)
MI	0.4698	0.0531	0.1622	0.0086	6.1971	75.24027
Bivariate Tree	47.6442	4.8756	16.1889	1.4981	1146.8028	18.10481

5. Conclusion

In this paper we developed a method based on the Heston model for the pricing of American options under stochastic volatility. There are various methods that resolve the pricing problem, so we decided to test the proposed finite difference scheme against other methods based on the Heston model too. We randomly chose the parameters for the American options, and we tested for put and call, having the Least Square Monte Carlo model as benchmark, to test the accuracy of the results.

We can see in the error tables that the method developed has the larger computing time, but on the other hand has the smallest errors. Except for the maximum relative error and the mean relative error in the put options, where the M.S. Approximation model has the smallest error.

Aspects that can be improved in the future having other model as benchmark, for example the Clarke and Parrott (1999) model where the error to the true price is smaller than the Least Square Monte Carlo model, as shown in the Rouah (2013) book, and computing the programs with more steps in their respectively grids.

References

[1] Heston, S. L. 1993. A Closed-form Solution for Options with Stochastic Volatility and Applications to Bond and Currency Options. *Review of Fiancial Studies* 6(2) 327-343.

[2] Broadie, M., Detemple, J. 1996. American Option Valuation: New Bounds, Approximations, and a Comparison of Existing Methods *Review of Fiancial Studies* 9(4) 1211-1250.

[3] Bakshi, G., Cao, C., Chen, Z. 1997. Empirical Performance of Alternative Option Pricing Models *The Journal of Finance* 52(5) 2003-2049.

[4] Clarke, N., Parrott, K. 1999. Multigrid for American Option Pricing with Stochastic Volatility *Applied Mathematical Finance* 6 177-195.

[5] Longstaff, F. A., Schwartz E. S. 2001. Valuing American Options by Simulation: A Simple Least-Squares Approach. *Review of Fiancial Studies* 14(1) 113-147.

[6] Detemple, J., Tian, W. 2002. The Valuation of American Options for a Class of Diffusion Processes *Management Science* 48(7) 917-937.

[7] Broadie, M., Detemple, J. 2004. Option Pricing: Valutaion Models and Applications *Management Science* 50(9) 1145-1177.

[8] Beliaeva, N. A., Nawalkha, S. K. 2010. A Simple Approach to Pricing American Options Under the Heston Stochastic Volatility Model *Journal of Derivatives* 17(1) 25-43.

[9] Medvedev, A., Scaillet, O. 2010. Pricing American Options under Stochastic Volatility and Stochastic Interest Rates *Journal of Financial Economics* 98(1) 145-159.

[10] Chockalingam, A., Muthuraman. K. 2011. American Options Under Stochastics Volatility. *Operations Research* 59 793-809.

[11] Rouah, F. D. 2013. The Heston Model and Its Extensions in Matlab and C#. *Wiley*

Appendix

```
function U = MHestonPDEUniformGrid(kappa,theta,sigma,v0,rho,lambda, ...
    K, r, q, S, V, T, PutCall, EuroAmer)

%Heston parameters
%kappa = params(1);
%theta = params(2);
%sigma = params(3);
%v0 = params(4);
%rho = params(5);
%lambda = params(6);

%Grid measurements

NS = length(S);
NV = length(V);
NT = length(T);
Smin = S(1); Smax = S(NS);
Vmin = V(1); Vmax = V(NV);
Tmin = T(1); Tmax = T(NT);
dt = (Tmax - Tmin)/(NT - 1);
dS = (Smax - Smin)/(NS - 1);
dV = (Vmax - Vmin)/(NV - 1);

%Initialize the 2-D grid with zeros
U = zeros(NS,NV);

%Temporary grid for previous time steps
u = zeros(NS,NV);

%Boundary condition fot t=maturity
for s=1:NS
    if strcmp(PutCall, 'C')
        U(s,:) = max(S(s) - K, 0);
    elseif strcmp(PutCall, 'P')
        U(s,:) = max(K - S(s), 0);
    end
end

%Go through the times
for t=1:NT-1
    %Boundary condition for Smin and Smax
    U(1,:) = 0;
    if strcmp(PutCall, 'C')
        U(NS,:) = max(0, Smax - K);
        U(:,NV) = max (0, S - K);
    elseif strcmp(PutCall, 'P')
        U(NS,:) = max(0, K - Smax);
        U(:,NV) = max (0, K - S);
    end

%Update the temporary grid u(s,t) with the boundary conditions
u = U;
```

```
%Boundary condition for Vmin
for s=2:NS-1
    LHS = u(s,1)*((-kappa*theta)/(V(2)-V(1)) - r) + ...
         (r-q)*(S(s)/2*dS)*(u(s+1,1) - u(s-1,1)) + ...
         (kappa*theta)/(V(2)-V(1)) * u(s,2);
    U(s,1) = LHS*dt + u(s,1);
end

%Update the temporary grid u(s,t) with the boundary conditions
u = U;

%Interior points of the grid (non boundary)
for s=2:NS-1
    for v=2:NV-1
        D1 = (0.5*S(s)^2*V(v)/dS^2 + (r-q)*0.5*S(s)/dS);
        D2 = (0.5*S(s)^2*V(v)/dS^2 - (r-q)*0.5*S(s)/dS);
        D3 = (0.5*sigma^2*V(v)/dV^2) + kappa*theta*0.5/dV - ...
            V(v)*kappa*0.5/dV;
        D4 = (0.5*sigma^2*V(v)/dV^2)    - kappa*theta*0.5/dV + ...
            V(v)*kappa*0.5/dV;
        D5 = (rho*sigma*S(s)*V(v))/(4*dV*dS);
        D6 = -V(v)*S(s)^2/dS^2 - (sigma^2*V(v))/dV^2 - r;

        L = u(s+1,v)*D1 + u(s-1,v)*D2 + u(s,v+1)*D3 + u(s,v-1)*D4 + ...
            (u(s+1,v+1) - u(s-1,v+1) - u(s+1,v-1) + u(s-1,v-1))*D5 ...
            + u(s,v)*D6;
        U(s,v) = L*dt + u(s,v);
    end
end

if strcmp (EuroAmer,'A')
    for s=1:NS
        if strcmp(PutCall, 'C')
            U(s,:) = max(U(s,:), S(s) - K);
        elseif strcmp(PutCall, 'P')
            U(s,:) = max(U(s,:), K - S(s));
        end
    end
end

end
```